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APPROXIMATE KERR-NEWMAN-LIKE METRIC WITH QUADRUPOLE

MÉTRICA APROXIMADA TIPO KERR-NEWMAN CON CUADRUPOLO

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Abstract

The Kerr metric is known to have issues when trying to find a *physical* interior solution. In this work we continue our efforts to construct a more realistic exterior metric for astrophysical objects. A new approximate metric representing the spacetime of a charged, rotating and slightly-deformed body is obtained by perturbing the Kerr-Newman metric to include the mass-quadrupole and quadrupole-quadrupole orders. It has a simple form because it is Kerr-Newman-like.

Keywords: general relativity; solutions of Einstein's equations; approximation procedures; weak fields.

Resumen

Se sabe que la métrica de Kerr tiene problemas al tratar de encontrar una solución física interior. En este trabajo continuamos nuestros esfuerzos para construir una métrica exterior más realista para describir objetos astrofísicos. Una nueva métrica aproximada que representa el espacio-tiempo de un cuerpo cargado, giratorio y ligeramente deformado, se obtiene perturbando la métrica de Kerr-Newman para incluir los órdenes de masa-cuadrupolo y cuadrupolo-cuadrupolo. Tiene una forma simple porque es similar a Kerr-Newman.

Palabras clave: relatividad general; soluciones de las ecuaciones de Einstein; procedimientos de aproximación; campos débiles.

Mathematics Subject Classification: 83C05, 83C25, 85-02.

1 Introduction

Since Kerr proposed his metric in 1963 [19] multiple efforts have been directed to finding an interior solution of his space-time [4, 14, 17, 21, 22, 24, 34, 35] or in generalizing the Kerr metric to a metric that allows a physical interior matching [15, 33, 31, 38]. Nonetheless, no physical interior solution exists. Even though the Kerr metric does not seem to have a physical interior solution, it has been widely successful in astrophysics and astronomy. Hence one should expect that a more realistic metric could be found by perturbing the Kerr space-time. See [1] for a relatively recent perspective in the issues present in the Kerr metric that complicate the interior matching.

Furthermore, from the ongoing efforts to construct an interior metric of the Kerr metric one can see a slight trend for preference of an oblate spheroid instead of a sphere [5, 39]. This is a mathematical motivation indicating the value of exploring metrics of deformed objects. However, the physical motivation is much more simple: real astrophysical objects are not perfect spheres, hence allowing for small deformations in a rotating object is a meaningful exercise.

Moreover, the interest in space-times capable of describing charged distributions has always been high. In the early 1916's Reissner and Nordström [29, 36] found their metric, which described a static spherically symmetric charge distribution. Even though this metric does not handle rotation, it has been used extensively in astronomy, *e.g.* for black hole lensing [6] and, in particular, to study Hawking radiation from a Reissner-Nordström black hole [40]. It is important to highlight that this kind of studies would benefit from a metric capable of describing charged objects but including rotation, and with the capabilities of allowing deformed objects. This last property is not fulfilled by the Kerr-Newman metric [27].

In this article we continue with our efforts in constructing a more realistic metric capable of representing a real astrophysical object. Here we use a perturbative method which utilizes the Lewis metric [2] in order to find space-times with quadrupole moment while using the Kerr space-time as a seed metric. Basically, our technique consists in cleverly changing the potentials of the Lewis metric while maintaining the cross term (rotational term). We have already applied this technique and obtained other approximate metrics [12, 11, 26]. In comparison to our previous efforts this work includes the addition of charge. Hence, our new metric is capable of representing a charged, rotating and slightly deformed massive object.

One of the main uncertainties when computing new solutions of the Einstein Maxwell Field Equations (EMFE) is how to prove that a given metric would have physical meaning[1]. In order to confirm the physical legitimacy of a given metric one can expand it to its post-linear from and compare the result with the post-linear version of the Hartle-Thorne (HT) metric [15, 31]. Note that it is possible to find an inner solution to the HT metric [1], and hence if a metric has a similar form to the HT metric then an inner solution should exists. See [13, 10] for a more detailed discussion and some examples.

[1]Here physical meaning stands for having an interior metric.

The rest of this paper is organized as follows. Our perturbation method of the Kerr metric with the help of the Lewis potentials is shown in section 2. In section 3, a new metric is obtained by means of our perturbative technique. This metric has rotation, quadrupole moment and charge. It is checked that the metric is a solution of the EMFE using a `REDUCE` code [16], this program is available upon request. The comparison of our metric with the HT one is presented in 4. In section 5, the Petrov type of the metric is found. It is type I like the Erez-Rosen metric. We conclude in section 6.

2 The perturbing method for the Kerr metric

Here, and in the following section we will use the method developed by Frutos et al. in [12, 11, 26] to obtain a Kerr-Newman-like metric, *i.e.* a space-time capable of describing a slightly deformed rotating charged mass. We start by using the connection between the Lewis metric and the Kerr metric to obtain the Lewis potentials associated to our new metric. These potentials are later used in the perturbation method.

First of all, we start with the Lewis metric, which is given by [2]

$$ds^2 = -V dt^2 + 2W dt d\phi + X d\rho^2 + Y dz^2 + Z d\phi^2, \tag{1}$$

where the chosen canonical coordinates are $x^1 = \rho$ and $x^2 = z$. The potentials V, W, Z, $X = \mathrm{e}^{\mu}$ and $Y = \mathrm{e}^{\nu}$ are functions of ρ and z with $\rho^2 = VZ + W^2$.

From [2] the transformation that leads to the Kerr metric is

$$\rho = \sqrt{\Delta}\sin\theta \qquad \text{and} \qquad z = (r - M)\cos\theta, \tag{2}$$

where $\Delta = r^2 - 2Mr + a^2 + e^2$, a is the rotational parameter and e is the electric charge.

Now, the Lewis potentials are chosen as follows

$$\begin{aligned}
V &= V_{KN}\,\mathrm{e}^{-2\psi} = \frac{1}{\tilde{\rho}^2}[\Delta - a^2\sin^2\theta]\,\mathrm{e}^{-2\psi},\\
W &= W_{KN} = -\frac{(2Jr - ae^2)}{\tilde{\rho}^2}\sin^2\theta,\\
X &= X_{KN}\mathrm{e}^{2\chi} = \tilde{\rho}^2\frac{\mathrm{e}^{2\chi}}{\Delta},\\
Y &= Y_{KN}\mathrm{e}^{2\chi} = \tilde{\rho}^2\,\mathrm{e}^{2\chi},\\
Z &= Z_{KN}\mathrm{e}^{2\psi} = \frac{\sin^2\theta}{\tilde{\rho}^2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\,\mathrm{e}^{2\psi},
\end{aligned} \tag{3}$$

where the potentials V_{KN}, W_{KN}, X_{KN}, Y_{KN}, Z_{KN} are the Lewis potentials for the Kerr-Newman metric, and $\tilde{\rho}^2 = r^2 + a^2\cos^2\theta$. Also, $J = Ma$ is the angular momentum.

The cross term potential W_{KN} is unaltered to preserve the following metric form

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\tilde{\rho}^2}[\mathrm{e}^{-\psi}dt - a\mathrm{e}^{\psi}\sin^2\theta d\phi]^2 + \frac{\sin^2\theta}{\tilde{\rho}^2}[(r^2+a^2)\mathrm{e}^{\psi}d\phi - a\mathrm{e}^{-\psi}dt]^2 \\ &+ \tilde{\rho}^2\mathrm{e}^{2\chi}\left(\frac{dr^2}{\Delta} + d\theta^2\right). \end{aligned} \tag{4}$$

These potentials guarantee that one gets the Kerr metric if $\psi = \chi = 0$. The function ψ and χ will be found approximately from the EMFE.

3 The approximate Kerr-Newman metric with quadrupole

As was stated in the previous section our problem has been reduced to finding the functions ψ and χ. Here we proceed to find such functions by solving the EMFE perturbatively. Also, we discuss the limiting cases of the new metric.

The EMFE are given by

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{R}{2}g_{ij} = \kappa T_{ij}, \\ \nabla_j F^{ij} &= \frac{1}{\sqrt{-g}}\partial_j[\sqrt{-g}F^{ij}] = 0, \end{aligned} \tag{5}$$

where G_{ij} $(i,\, j = 0,\, 1,\, 2,\, 3)$ are the Einstein tensor components, R_{ij} are the Ricci tensor components, R is the curvature scalar, $\kappa = 8\pi G/c^4$, $g = \det(g_{ij})$ and T_{ij} represent the energy-momentum tensor components, which are given by

$$4\pi T_{ij} = g^{kl}F_{il}F_{jk} - \frac{1}{4}F^{ab}F_{ab}\,g_{ij}, \tag{6}$$

where $F_{ij} = \partial_j A_i - \partial_i A_j$ are the electromagnetic tensor components, and A_i is the vector potential components.

The 1-form for the vector potential A can be written as follows

$$\mathsf{A} = -\frac{er}{\tilde{\rho}^2}\left[\mathrm{e}^{-\psi}dt - a\mathrm{e}^{\psi}\sin^2\theta d\phi\right]. \tag{7}$$

In addition, the 2-form for the electromagnetic tensor F can be obtained as follows

$$\mathsf{F} = d\mathsf{A} = \frac{1}{2} F_{ij} dx^j \wedge dx^j. \tag{8}$$

From (8) we can determine the energy-momentum tensor.

Let us highlight the terms that are neglected in our perturbative approach, which are

$$\begin{aligned} W^2 \frac{\partial \psi}{\partial x^i} &\sim 0, \\ W \frac{\partial W}{\partial x^i} \frac{\partial \psi}{\partial x^i} &\sim 0, \\ W^2 \frac{\partial \chi}{\partial x^i} &\sim 0, \\ W \frac{\partial W}{\partial x^i} \frac{\partial \chi}{\partial x^i} &\sim 0. \end{aligned}$$

Moreover, eliminating the terms corresponding to the Kerr metric in the Ricci tensor components, we get the Ricci tensor component of the appendix of [11]. Note that W plays the role of the rotation, since it is proportional to the angular momentum. However, the above expressions do not mean that factors of J^2 or a^2 vanish, what is being effectively restricted here are combinations of quadrupoles with the angular momentum or the rotation.

In order to obtain an expression for the Ricci tensor and the curvature scalar, an Ansatz for ψ and χ with coefficients is proposed [10]. After solving (5), one finds the Ansatz coefficients:

$$\begin{aligned} \psi &= \frac{q}{r^3} P_2 + 3 \frac{Mq}{r^4} P_2, \qquad (9) \\ \chi &= \frac{qP_2}{r^3} + \frac{1}{3} \frac{Mq}{r^4} (-1 + 5P_2 + 5P_2^2) + \frac{1}{9} \frac{q^2}{r^6} (2 - 6P_2 - 21P_2^2 + 25P_2^3), \end{aligned}$$

where q represents the quadrupole parameter and P_2 is the usual Legendre polynomial, $P_2 = (3\cos^2\theta - 1)/2$.

Finally we have all the information we require to construct our Kerr-Newman-like metric. From (4), the metric components are given by

$$\begin{aligned} g_{tt} &= \frac{\mathrm{e}^{-2\psi}}{\tilde{\rho}^2}[a^2\sin^2\theta - \Delta], \\ g_{t\phi} &= \frac{a}{\tilde{\rho}^2}[\Delta - (r^2+a^2)]\sin^2\theta = \frac{\sin^2\theta}{\tilde{\rho}^2}(ae^2 - 2Jr), \\ g_{rr} &= \tilde{\rho}^2\frac{\mathrm{e}^{2\chi}}{\Delta}, \\ g_{\theta\theta} &= \tilde{\rho}^2\mathrm{e}^{2\chi}, \\ g_{\phi\phi} &= \frac{\mathrm{e}^{2\psi}}{\tilde{\rho}^2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\sin^2\theta. \end{aligned} \tag{10}$$

We checked that (10) was valid up to order $O(aq^2,\, a^2q,\, Maq,\, Mq^2,\, M^2q,\, e^2q,\, q^3)$ using a `REDUCE` code [16].

Now we will focus on the limiting cases of the new metric, *i.e.* equation (10). We summarized the limiting cases in Table 1. First, note that if $e=0$ in (10) one recovers the metric found by Frutos *et al.* in [11]. Therefore, following [11] the other interesting limiting cases are: The Kerr metric if $e=q=0$, the metric found in [12] if $e=a^2=q^2=0$, the Erez-Rosen-like metric described in [11] if $a=e=0$, and the Schwarzchild metric if $a=e=q=0$. Furthermore, one obtains the Kerr-Newman geometry if $q=0$. Also, the Reissner-Nordström metric is obtained if $a=q=0$. Thus, all the expected limiting cases can be obtained from this new metric.

Table 1: Limiting cases.

Absent physical property	Small physical property	Limiting metric
Charge	Quadrupole (linear)	Metric found in [11]
Charge	Quadrupole (quadratic)	Metric found in [10]
Quadrupole	-	Kerr-Newman
Quadrupole and rotation	-	Reissner-Nordström
Charge and quadrupole	-	Kerr
Charge and rotation	-	Erez-Rosen-like
Charge	Quadrupole and rotation	Metric found in [12]
Charge, quadrupole and rotation	-	Schwarzchild

Here we will not show the matching of (10) with the HT metric. This matching is vital because it guarantees that an interior solution exists. We will not show it here, since it should be trivial to follow [11] because our metric has the same form, or equivalently one could follow [13].

Moreover, in [10], it was shown that the multipole structure of this metric without charge is non-isometric with the Quevedo-Mashhoon [33, 31] and the Manko-Novikov [25] metrics. Then, this new metric should not be isometric with charged version of these metrics.

In [10], the first ten relativistic multipoles for this metric without electric charge using the procedure described in [9] were found. Due to the fact, that our metric contains the Reissner-Nordström and the Kerr-Newman spacetimes, it is not necessary to find the relativistic electromagnetic multipoles, but in the paper of Hoenselaers and Perjés [18] one can find the algorithm to determine them. Sotiriou and Apostolatos [37] corrected some erros of the expressions given by [18]. Furthermore, they obtained the relativistic electromagnetic multipoles for the Kerr-Newman metric [37].

4 Comparison to the Hartle-Thorne metric

In this section, we show that it is possible to transform our metric to the HT one. To do so, we obtain the post-linear forms of these metrics (HT and our metric). The post-linear form of our metric is

$$
\begin{aligned}
g_{tt} &= -\left(1 - 2\frac{M}{r} + 2\frac{Ma^2}{r^3}\cos^2\theta - 2\frac{q}{r^3}P_2 - 2\frac{Mq}{r^4}P_2\right), \\
g_{t\phi} &= -2\frac{J}{r}\sin^2\theta, \\
g_{rr} &= 1 + 2\frac{M}{r} + 4\frac{M^2}{r^2} - \frac{a^2}{r^2}\sin^2\theta - 2\frac{Ma^2}{r^3}(1+\sin^2\theta), \\
&- 4\frac{M^2a^2}{r^4}(2+\sin^2\theta) + 2\frac{q}{r^3}P_2 + \frac{2}{3}\frac{Mq}{r^4}(5P_2^2 + 11P_2 - 1), \\
g_{\theta\theta} &= r^2\left(1 + \frac{a^2}{r^2}\cos^2\theta + 2\frac{q}{r^3}P_2 + \frac{2}{3}\frac{Mq}{r^4}(5P_2^2 + 5P_2 - 1)\right), \\
g_{\phi\phi} &= r^2\sin^2\theta\left(1 + \frac{a^2}{r^2} + 2\frac{Ma^2}{r^3}\sin^2\theta + 2\frac{q}{r^3}P_2 + 6\frac{Mq}{r^4}P_2\right),
\end{aligned}
\tag{11}
$$

where the second order in q is omitted and $e = 0$ in order to compare with HT.

The post-linear form of the HT metric is

$$
\begin{aligned}
g_{tt} &= -\left(1 - 2U + 2\frac{q}{r^3}P_2 + 2\frac{Mq}{r^4}P_2 - \frac{2}{3}\frac{J^2}{r^4}(2P_2+1)\right), \\
g_{t\phi} &= -2\frac{J}{r}\sin^2\theta, \\
g_{rr} &= 1 + 2U + 4U^2 - 2\frac{q}{r^3}P_2 - 10\frac{Mq}{r^4}P_2 + 2\frac{J^2}{r^4}(8P_2 - 1), \\
g_{\theta\theta} &= r^2\left(1 - 2\frac{q}{r^3}P_2 - 5\frac{Mq}{r^4}P_2 + \frac{J^2}{r^4}P_2\right), \\
g_{\phi\phi} &= r^2\sin^2\theta\left(1 - 2\frac{q}{r^3}P_2 - 5\frac{Mq}{r^4}P_2 + \frac{J^2}{r^4}P_2\right).
\end{aligned}
\tag{12}
$$

The transformation from our metric to the HT one can be obtained from [10], and changing $q \to Ma^2 - q$, at the same level of approximation:

$$
\begin{aligned}
r &= R\left[1 + \frac{Mq}{R^4}f_1 + \frac{a^2}{R^2}\left(h_1 + \frac{M}{R}h_2 + \frac{M^2}{R^2}h_3\right)\right], \\
\theta &= \Theta + \frac{Mq}{R^4}f_2 + \frac{a^2}{R^2}\left(h_4 + \frac{M}{R}h_5\right),
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
f_1 &= -\frac{1}{9}(1 + 4P_2 - 5P_2^2), \\
f_2 &= \frac{1}{6}(2 - 5P_2)\cos\Theta\sin\Theta, \\
h_1 &= -\frac{1}{2}\sin^2\Theta, \\
h_2 &= -\frac{1}{2}\sin^2\Theta, \\
h_3 &= 1 - 3\cos^2\Theta = -2P_2, \\
h_4 &= -\frac{1}{2}\cos\Theta\sin\Theta, \\
h_5 &= -\cos\Theta\sin\Theta.
\end{aligned}
$$

Since our post-linear metric can be transformed to the HT spacetime, it is possible to construct an interior metric that could be matched to our exterior spacetime.

5 Petrov classification

To classify gravitational fields, Petrov devised a method based on the algebraic symmetries of the Weyl tensor at each event in a Lorentzian manifold [30, 32]. It is customary to employ tetrads to determine the Petrov type. In the Newman-Penrose formalism [28], the metric components can be written as follows

$$g^{ij} = l^i n^j + l^j n^i - (m^i \bar{m}^j + m^j \bar{m}^i), \tag{14}$$

where l^i, n^j, m^i ($\bar{m}^i$ is the complex conjugate of m^i) are the null tetrads. These tetrads has to fulfill

$$l_i l^i = n_i n^i = m_i m^i = \bar{m}_i \bar{m}^i = l_i m^i = l_i \bar{m}^i = n_i m^i = n_i \bar{m}^i = 0,$$

and

$$l_i n^i = n_i l^i = -1, \qquad m_i \bar{m}^i = \bar{m}_i m^i = 1.$$

In our case, the tetrads can be easily constructed from the Kinnersley tetrads for the Kerr-Newman metric [20].

$$\begin{aligned}
l^j &= \left(\frac{\mathrm{e}^{\psi}}{\Delta}(r^2+a^2),\, \mathrm{e}^{-\chi}, 0,\, \frac{a}{\Delta}\mathrm{e}^{-\psi}\right), \\
n^j &= \left(\frac{\mathrm{e}^{\psi}}{2\rho^2}(r^2+a^2),\, -\frac{\Delta}{2\rho^2}\mathrm{e}^{-\chi},\, 0,\, \frac{a}{2\rho^2}\mathrm{e}^{-\psi}\right), \\
m^j &= \left(i\frac{a\mathrm{e}^{\psi}\sin\theta}{\sqrt{2}\rho^2},\, 0,\, \frac{\mathrm{e}^{-\chi}}{\sqrt{2}\rho^2},\, i\frac{\mathrm{e}^{-\psi}}{\sqrt{2}\rho^2\sin\theta}\right)(r - ia\cos\theta), \\
\bar{m}^j &= \left(-i\frac{a\mathrm{e}^{\psi}\sin\theta}{\sqrt{2}\rho^2},\, 0,\, \frac{\mathrm{e}^{-\chi}}{\sqrt{2}\rho^2},\, -i\frac{\mathrm{e}^{-\psi}}{\sqrt{2}\rho^2\sin\theta}\right)(r + ia\cos\theta).
\end{aligned} \tag{15}$$

From (15), the five Newman-Penrose (NP) scalars[2] are determined.

$$\begin{aligned}
\Psi_0 &= W_{abcd} l^a m^b l^c m^d, \\
\Psi_1 &= W_{abcd} l^a n^b l^c m^d, \\
\Psi_2 &= W_{abcd} l^a m^b \bar{m}^c n^d, \\
\Psi_3 &= W_{abcd} l^a n^b \bar{m}^c n^d, \\
\Psi_4 &= W_{abcd} n^a \bar{m}^b n^c \bar{m}^d,
\end{aligned} \tag{16}$$

[2]Note that in the NP paper, all scalars are negative.

where W_{ijlm} are the Weyl tensor components,

$$\begin{aligned} W_{ijlm} &= R_{ijlm} + \frac{1}{2}(g_{im}R_{lj} + g_{jl}R_{mi} - g_{il}R_{mj} - g_{jm}R_{li}) \\ &+ \frac{R}{6}(g_{il}g_{mj} - g_{im}g_{lj}). \end{aligned} \tag{17}$$

To determine the Petrov type, the multiplicity of roots of the following quartic equation in the complex variable z has to be found

$$\Psi_0 z^4 + 4\Psi_1 z^3 + 6\Psi_2 z^2 + 4\Psi_3 z + \Psi_4 = 0. \tag{18}$$

In Table 2, we explain the different categories of the Petrov classification.

Table 2: Petrov Classification.

Petrov type	Solution of (18)
Type I	Four distinct roots
Type II	One root of multiplicity 2 and the other two roots distinct
Type III	One root of multiplicity 3 and the other root distinct
Type D	Two distinct roots of multiplicity 2
Type N	One root of multiplicity 4
Type O	All $\Psi_i = 0$, degenerate

Using the Weyl tensor components obtained by means of a `REDUCE` code, and the tetrads (15), these scalars become

$$\begin{aligned} \Psi_0 &= -\frac{3qr^3\sin^2\theta}{\rho^6\Delta}\mathrm{e}^{-2\chi}, \\ \Psi_1 &= \frac{12qr^3\sin\theta\cos\theta}{\sqrt{2}\rho^6\Delta}\mathrm{e}^{-2\chi}, \\ \Psi_2 &= \left[\Psi_{2KN} - \frac{6qrP_2}{\rho^6}\right]\mathrm{e}^{-2\chi}, \\ \Psi_3 &= -\frac{6qr\sin\theta\cos\theta}{\sqrt{2}\rho^6}\mathrm{e}^{-2\chi}, \\ \Psi_4 &= -\frac{3qr\Delta\sin^2\theta}{4\rho^2}\mathrm{e}^{-2\chi}, \end{aligned} \tag{19}$$

where

$$\Psi_{2KN} = \frac{1}{(r - ia\cos\theta)^3}\left[-M + \frac{e^2}{(r + ia\cos\theta)}\right]. \tag{20}$$

When q vanishes, we recover the scalars for Kerr-Newman [3]. Following the algorithm of Letniowski and McLenaghan [23], one has to calculate the following additional scalars to find the Petrov type

$$\begin{aligned} \mathcal{H} &= \Psi_0\Psi_2 - \Psi_1^2 \neq 0, \\ \mathcal{I} &= \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2 \neq 0, \\ \mathcal{G} &= \Psi_0^2\Psi_3 - 3\Psi_0\Psi_1\Psi_2 + 2\Psi_1^3 \neq 0, \\ \mathcal{Z} &= \Psi_0^2\mathcal{I} - 12\mathcal{H}^2 \neq 0. \end{aligned} \tag{21}$$

From these scalars and following this algorithm, it is found that the metric is type I up to order $O(aq^2,\, a^2q,\, Maq,\, Mq^2,\, M^2q,\, e^2q,\, eq^2\, q^3)$, except on the symmetry axis, where it is exactly type D. If one neglects the q^2 terms, the metric approaches the type D. This behavior is similar as in the Erez-Rosen metric [32].

6 Conclusion

A metric with charge, deformations and rotation was obtained by solving the EMFE perturbatively. The expected limiting cases of this new metric were recovered. Moreover, these limiting cases confirm that our metric adequately describes a mass with charge and quadrupole under rotation. We successfully applied the perturbation method developed by Frutos et al in [12, 11, 10, 26] to obtain a new approximate metric. Notice that the main improvement of our work with respect to [11, 10] is the inclusion of charge. Ideally, this moves us closer to representing an actual astrophysical object. Further, one could expect that careful understanding of this procedure might eventually lead us towards a magnetized object.

We showed that the new metric can be matched to the HT metric, as in the previous cases of the non-charged metrics [11, 10]. Particularly, because of the similarity with that non-charged metric we do not expect any issues in the matching. This matching is important for the sake of showing that an interior metric of the new metric exists. Moreover, a charged version of HT can be obtained from our metric easily.

With respect to the Petrov classification, we found that the new solution is type I up to order third order. It presents a similar behavior as the Erez-Rosen metric.

Since this new metric can describe real charged astrophysical objects in a more realistic fashion than the Kerr-Newman or Reissner-Nordström metrics, our metric should be attractive for astrophysical applications, like gravitational lensing and relativistic magnetohydrodynamics.

Furthermore, computational implementation of this metric should not imply additional difficulties with respect to current methodologies, since it maintains a similar form to the commonly used Kerr metric.

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