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SLOWLY ROTATING CURZON-CHAZY METRIC

MÉTRICA DE CURZON-CHAZY CON ROTACIÓN LENTA

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Abstract

A new rotation version of the Curzon-Chazy metric is found. This new metric was obtained by means of a perturbation method, in order to include slow rotation. The solution is then proved to fulfill the Einstein's equations using a REDUCE program. Furthermore, the applications of this new solution are discussed.

Keywords: general relativity; solutions of Einstein's equations; approximation procedures; weak fields.

Resumen

Se encontró una nueva versión rotante de la métrica de Curzon-Chazy. Esta nueva métrica fue obtenida por medio de un método perturbativo para incluir rotación lenta. Se prueba que la métrica obtenida es solución a las ecuaciones de Einstein por medio de un programa en REDUCE. Finalmente, se discuten las aplicaciones de esta nueva solución.

Palabras clave: relatividad general; soluciones de las ecuaciones de Einstein; procedimientos de aproximación; campos débiles.

Mathematics Subject Classification: 83C05, 83C20, 83C25.

1 Introduction

The Curzon-Chazy metric [6, 7, 19] is one of the simplest solutions of the Einstein field equations (EFE) for the Weyl metric. The original idea of Curzon [7] and Chazy [6] was the superposition of two particles at different points on the symmetry axis. This superposition exhibit a singularity between these particles along this symmetry axis. This singularity is interpreted as a strut (Weyl strut), which stress holds these particles apart and does not exert a gravitational field [3, 11].

For one particle, the Curzon-Chazy metric describes the exterior field of a finite source [11], and has a spherically symmetric Newtonian potential of a point particle located at $r = 0$. The resulting spacetime is not spherically symmetric and its weak limit is that of an object located at the origin with multipoles. Moreover, the singularity at $r = 0$ has a very interesting but complicated directionally dependent structure [10, 11]. There is a curvature singularity at $\rho = 0$, $z = 0$ that is not surrounded by a horizon, *i.e.* it is naked. Every light ray emitted from it becomes infinitely redshifted, so that it is effectively invisible [11]. Studying the principal null directions, it was found that this spacetime has an invariantly hypersurface $\sqrt{\rho^2 + z^2} = M$, on which the Weyl invariant $\mathcal{J}$ (determinant of the Weyl five complex scalar functions) vanishes [2, 11, 18]. Furthermore, this metric is Petrov type D, except at two points ($z = \pm M$) that intersect the axis $\rho = 0$, where it is of Petrov type O [2].

The properties of this solution have been analyzed since its discovery such as the potential surfaces for its time-like geodesics and their variations with the change in energy [9]. A generalization of the metric to the Einstein-Maxwell equations have also been obtained [15]. A modified Curzon-Chazy metric, using a rotating reference frame as approach, has already been applied to study binary pulsar systems [20].

A slowly rotating version of the Curzon-Chazy solution could be used, for instance, to study the gravitational lens effect, because of its asymmetrical nature. Furthermore, it is important to mention Halilsoy's research [12], in his work he obtained a rotating Curzon metric using the Ernst potential method [8], however the rotational metric component term (g_{03}) obtained in his work is non flat, therefore it could not be of astrophysical interest. Besides there is a problem with the generalization Halilsoy proposes for an arbitrary number of massive particules, it fails to obtain the correct form for the case of two massive particles.

In this work, a slow rotating metric is obtained by introducing a perturbation in the metric rather than using a rotating reference frame. Our rotational metric term is Kerr like [16]. Moreover, we discuss the possible applications of this new version.

2 The Curzon-Chazy metric

The Curzon-Chazy metric in canonical cylindrical coordinates is given by [4, 6, 7] (in geometrical units $G = c = 1$):

$$ds^2 = \mathrm{e}^{-2\psi} dt^2 - \mathrm{e}^{2(\psi-\gamma)}(d\rho^2 + dz^2) - \mathrm{e}^{2\psi}\rho^2 d\phi^2, \quad (1)$$

where

$$\begin{aligned} \psi &= \frac{M}{\eta}, \\ \gamma &= \frac{M^2\rho^2}{2\eta^4}, \\ \eta^2 &= \rho^2 + z^2. \end{aligned} \quad (2)$$

The Curzon-Chazy metric in spherical coordinates can be obtained by means of the following mapping [4, 5]:

$$\rho = \sqrt{Z} \quad \text{and} \quad z = (r - M)\cos\theta, \tag{3}$$

where $Z = (r^2 - 2Mr)\sin^2\theta$.

Using this transformation the Curzon-Chazy metric takes the form

$$ds^2 = \mathrm{e}^{-2\psi}dt^2 - \mathrm{e}^{2(\psi-\gamma)}(Xdr^2 + Yd\theta^2) - \mathrm{e}^{2\psi}Zd\phi^2, \tag{4}$$

where

$$\begin{aligned} \psi &= \frac{M}{\eta}, \\ \gamma &= \frac{M^2(r^2-2Mr)\sin^2\theta}{2\eta^4}, \\ \eta^2 &= r^2 - 2Mr + M^2\cos^2\theta \\ \Delta &= r^2 - 2Mr + M^2\sin^2\theta, \\ X &= \frac{\Delta}{r^2-2Mr}, \\ Y &= \Delta. \end{aligned} \tag{5}$$

3 The Lewis metric

Now, we need a new metric to include a new feature (rotation) into a given seed metric, i.e. the Curzon-Chazy spacetime. The Lewis metric is given by [17, 4]

$$ds^2 = Vdt^2 - 2Wdtd\phi - \mathrm{e}^{\mu}d\rho^2 - \mathrm{e}^{\nu}dz^2 - Qd\phi^2, \tag{6}$$

where we have chosen the canonical coordinates $x^1 = \rho$ and $x^2 = z$. The potentials V, W, X, μ and ν are functions of ρ and z ($\rho^2 = VQ + W^2$). Choosing $\mu = \nu$ and performing the following changes of potentials

$$V = f, \qquad W = \omega f, \qquad Q = \frac{\rho^2}{f} - \omega^2 f, \qquad \mathrm{e}^{\mu} = \frac{\mathrm{e}^{\chi}}{f}, \tag{7}$$

we get the Papapetrou metric

$$ds^2 = f(dt - \omega d\phi)^2 - \frac{\mathrm{e}^{\chi}}{f}[d\rho^2 + dz^2] - \frac{\rho^2}{f}d\phi^2. \tag{8}$$

Note that for slow rotation we neglect the second order in ω, hence $\omega^2 \simeq 0 \Rightarrow W^2 \simeq 0$, and $Q \simeq \rho^2/f$.

4 Perturbing the Curzon-Chazy metric

To include slow rotation into the Curzon-Chazy metric we use the following procedure:

- choose expressions for the canonical coordinates ρ and z (in this case equations (3)),
- choose V, Q and μ (with $\mu = \nu$ and $\omega^2 \simeq 0 \Rightarrow W^2 \simeq 0$) in such a way you can get the seed metric if it is not perturbed, and
- solve the EFE for the Lewis metric, equation (6) with the chosen potentials V, Q and μ, i.e. find W.

Let us apply this procedure for the Curzon-Chazy metric as seed metric. First of all, we choose expressions for the canonical coordinates ρ and z. From (3) we get

$$d\rho^2 + dz^2 = \Delta\left(\frac{dr^2}{r^2 - 2Mr} + d\theta^2\right) = Xdr^2 + Yd\theta^2, \tag{9}$$

where

$$X = \frac{\Delta}{r^2 - 2Mr} \quad \text{and} \quad Y = \Delta.$$

Now, let us choose

$$V = f = \mathrm{e}^{-2\psi} \quad \text{and} \quad \mathrm{e}^{\mu} = \frac{\mathrm{e}^{\chi}}{f} = \mathrm{e}^{2(\psi-\gamma)},$$

and neglecting the second order in W, we have

$$Q \simeq \frac{\rho^2}{f} = Z\mathrm{e}^{2\psi} = \mathrm{e}^{2\psi}(r^2 - 2Mr)\sin^2\theta.$$

The Lewis metric takes the form

$$ds^2 = \mathrm{e}^{-2\psi}dt^2 - 2W dt d\phi - \mathrm{e}^{2(\psi-\gamma)}[Xdr^2 + Yd\theta^2] - Z\mathrm{e}^{2\psi}d\phi^2. \tag{10}$$

To obtain a slowly rotating version of the metric (4), the only potential, we have to find is W. In order to do this, we need to solve the EFE for this metric:

$$G_{ij} = R_{ij} - \frac{R}{2}g_{ij} = 0, \tag{11}$$

where $R_{ij}\ (i,\ j = 0,\ 1,\ 2,\ 3)$ are the Ricci tensor components and R is the curvature scalar. To find the approximated slowly version of the metric, we wrote a REDUCE program to find the Ricci tensor [14]. The interested reader can request this program by sending us a message. Fortunately, the Ricci tensor components R_{00}, R_{11}, R_{12}, R_{22}, R_{23}, R_{33} and the curvature scalar depend upon the potentials V, X, Y, Z and not on W (see Appendix). Hence, these components vanish. The only equation we have to solve is $R_{03} = 0$, because it depends upon W. The equation for this Ricci component, up to order $O(M^3, a^2)$, is

$$\sin\theta\left(\frac{\partial^2 W}{\partial\theta^2} + r^2\frac{\partial^2 W}{\partial r^2}\right) - \cos\theta\frac{\partial W}{\partial\theta} = 0. \tag{12}$$

The solution for (12) is

$$W = \frac{K}{r}\sin^2\theta. \tag{13}$$

In order to find the constant K let us see the Lense-Thirring metric which is obtained from the Kerr metric:

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4J}{r}\sin^2\theta dt d\phi - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2 d\Sigma^2, \tag{14}$$

where $d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ and $J = Ma$ is the angular momentum. At first order in M, this metric and the perturbed Curzon-Chazy metric coincide. Then, comparing the second term of the latter metric with the corresponding term of (14), we note that $K = -2J = -2Ma$.

Then, the slow rotating Curzon-Chazy metric is

$$\begin{aligned} ds^2 &= \mathrm{e}^{-2\psi}dt^2 + \frac{4J}{r}\sin^2\theta dt d\phi - \Delta\mathrm{e}^{2(\psi-\gamma)}\left(\frac{dr^2}{r^2 - 2Mr} + d\theta^2\right) \\ &- \mathrm{e}^{2\psi}(r^2 - 2Mr)\sin^2\theta d\phi^2. \end{aligned} \tag{15}$$

We check that the metric (15) is indeed a solution of the EFE, up to the order $O(M^3, a^2)$, using the same REDUCE program.

It is interesting to expand the metric (15) in a Taylor series to see its weak limit structure, the result is

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r} + \frac{2}{3}\frac{M^3}{r^3}P_2(\cos\theta)\right)dt^2 + \frac{4J}{r}\sin^2\theta dt d\phi \\ &- \left(1 + \frac{2M}{r} + \frac{4M^2}{r^2} + 2\left(4 - \frac{1}{3}P_2(\cos\theta)\right)\frac{M^3}{r^3}\right)dr^2 \\ &- r^2\left(1 - \frac{2}{3}\frac{M^3}{r^3}P_2(\cos\theta)\right)d\Sigma^2, \end{aligned} \tag{16}$$

where $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$ and $d\Sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$.

From a direct comparison with the Hartle-Thorne metric [13], we see that this metric could correspond to a spacetime of a massive object with mass quadrupole given by $\mathcal{Q} = M^3/3$.

5 Conclusion

The slow rotation version of the Curzon-Chazy metric was generated perturbing the Einstein equations from the Lewis metric. Slow rotating solutions are more realistic metrics that static metrics as the Curzon-Chazy spacetime. Moreover, Halilsoy found the first rotating version of this metric [12], but it is non flat. Due to this problem, this metric should not be used for astrophysical calculations. Our slow rotating version of the Curzon-Chazy spacetime is asymptotically flat and it could be applied to astrophysical systems where this spacetime should appear.

This new approximate metric has multiple applications. For instance, it could be used to describe binary systems or ring like systems [20, 1]. New calculations, such as the geodesics can be performed in order to visualize the trajectories due to such gravitational field. Also, a description of the surface potentials due to these geodesics can be studied, as it was done for the non-rotating Curzon-Chazy solution [9]. Moreover, for this spacetime, gravitational lens calculations can be performed, it would lead to a more real scenario for treating binary or ring-like systems.

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A Appendix

The Ricci tensor components are

$$\begin{aligned}
R_{00} &= \frac{e^{2(\gamma-2\psi)}}{2X^2Y^2Z}\left(-2X^2YZ\frac{\partial^2\psi}{\partial\theta^2} - XYZ\frac{\partial\psi}{\partial\theta}\frac{\partial X}{\partial\theta} + X^2Z\frac{\partial\psi}{\partial\theta}\frac{\partial Y}{\partial\theta}\right.\\
&- X^2Y\frac{\partial\psi}{\partial\theta}\frac{\partial Z}{\partial\theta} - 2XY^2Z\frac{\partial^2\psi}{\partial r^2} + Y^2Z\frac{\partial\psi}{\partial r}\frac{\partial X}{\partial r} - XYZ\frac{\partial\psi}{\partial r}\frac{\partial Y}{\partial r}\\
&\left.- XY^2\frac{\partial\psi}{\partial r}\frac{\partial Z}{\partial r}\right)\\
R_{01} &= 0\\
R_{02} &= 0\\
R_{03} &= \frac{e^{2(\gamma-\psi)}}{2X^2Y^2Z}\left(-XYZ\frac{\partial X}{\partial\theta}\frac{\partial W}{\partial\theta} + Y^2Z\frac{\partial X}{\partial r}\frac{\partial W}{\partial r} - 2X^2YZ\frac{\partial^2 W}{\partial\theta^2}\right.\\
&+ X^2Z\frac{\partial W}{\partial\theta}\frac{\partial Y}{\partial\theta} + X^2Y\frac{\partial W}{\partial\theta}\frac{\partial Z}{\partial\theta} - 2XY^2Z\frac{\partial^2 W}{\partial r^2} - XYZ\frac{\partial W}{\partial r}\frac{\partial Y}{\partial r}\\
&\left.+ XY^2\frac{\partial W}{\partial r}\frac{\partial Z}{\partial r}\right)\\
R_{11} &= \frac{1}{4XY^2Z^2}\left(-4X^2YZ^2\frac{\partial^2\psi}{\partial\theta^2} - 2XYZ^2\frac{\partial\psi}{\partial\theta}\frac{\partial X}{\partial\theta} + 2X^2Z^2\frac{\partial\psi}{\partial\theta}\frac{\partial Y}{\partial\theta}\right.\\
&- 2X^2YZ\frac{\partial\psi}{\partial\theta}\frac{\partial Z}{\partial\theta} - 4XY^2Z^2\frac{\partial^2\psi}{\partial r^2} - 8XY^2Z^2\left[\frac{\partial\psi}{\partial r}\right]^2 + 2Y^2Z^2\frac{\partial\psi}{\partial r}\frac{\partial X}{\partial r}\\
&- 2XYZ^2\frac{\partial\psi}{\partial r}\frac{\partial Y}{\partial r} - 2XY^2Z\frac{\partial\psi}{\partial r}\frac{\partial Z}{\partial r} + 4X^2YZ^2\frac{\partial^2\gamma}{\partial\theta^2} + 2XYZ^2\frac{\partial\gamma}{\partial\theta}\frac{\partial X}{\partial\theta}\\
&- 2X^2Z^2\frac{\partial\gamma}{\partial\theta}\frac{\partial Y}{\partial\theta} + 2X^2YZ\frac{\partial\gamma}{\partial\theta}\frac{\partial Z}{\partial\theta} + 4XY^2Z^2\frac{\partial^2\gamma}{\partial r^2} - 2Y^2Z^2\frac{\partial\gamma}{\partial r}\frac{\partial X}{\partial r}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2XYZ^2\frac{\partial\gamma}{\partial r}\frac{\partial Y}{\partial r} - 2XY^2Z\frac{\partial\gamma}{\partial r}\frac{\partial Z}{\partial r} - 2XYZ^2\frac{\partial^2 X}{\partial\theta^2} + YZ^2\left[\frac{\partial X}{\partial\theta}\right]^2 \\
&+ \quad XZ^2\frac{\partial X}{\partial\theta}\frac{\partial Y}{\partial\theta} - XYZ\frac{\partial X}{\partial\theta}\frac{\partial Z}{\partial\theta} + YZ^2\frac{\partial X}{\partial r}\frac{\partial Y}{\partial r} + Y^2Z\frac{\partial X}{\partial r}\frac{\partial Z}{\partial r} \\
&- \quad 2XYZ^2\frac{\partial^2 Y}{\partial r^2} + XZ^2\left[\frac{\partial Y}{\partial r}\right]^2 - 2XY^2Z\frac{\partial^2 Z}{\partial r^2} + XY^2\left[\frac{\partial Z}{\partial r}\right]^2\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
R_{12} &= \frac{1}{4XYZ^2}\Bigg(-8XYZ^2\frac{\partial\psi}{\partial\theta}\frac{\partial\psi}{\partial r} - 2XYZ\frac{\partial\gamma}{\partial\theta}\frac{\partial Z}{\partial r} - 2XYZ\frac{\partial\gamma}{\partial r}\frac{\partial Z}{\partial\theta} \\
&+ \quad YZ\frac{\partial X}{\partial\theta}\frac{\partial Z}{\partial r} + XZ\frac{\partial Y}{\partial r}\frac{\partial Z}{\partial\theta} - 2XYZ\frac{\partial^2 Z}{\partial\theta\partial r} + XY\frac{\partial Z}{\partial\theta}\frac{\partial Z}{\partial r}\Bigg) \\
R_{13} &= 0 \\
R_{22} &= \frac{1}{4X^2YZ^2}\Bigg(-4X^2YZ^2\frac{\partial^2\psi}{\partial\theta^2} - 8X^2YZ^2\left[\frac{\partial\psi}{\partial\theta}\right]^2 - 2XYZ^2\frac{\partial\psi}{\partial\theta}\frac{\partial X}{\partial\theta} \\
&+ \quad 2X^2Z^2\frac{\partial\psi}{\partial\theta}\frac{\partial Y}{\partial\theta} - 2X^2YZ\frac{\partial\psi}{\partial\theta}\frac{\partial Z}{\partial\theta} - 4XY^2Z^2\frac{\partial^2\psi}{\partial r^2} + 2Y^2Z^2\frac{\partial\psi}{\partial r}\frac{\partial X}{\partial r} \\
&- \quad 2XYZ^2\frac{\partial\psi}{\partial r}\frac{\partial Y}{\partial r} - 2XY^2Z\frac{\partial\psi}{\partial r}\frac{\partial Z}{\partial r} + 4X^2YZ^2\frac{\partial^2\gamma}{\partial\theta^2} + 2XYZ^2\frac{\partial\gamma}{\partial\theta}\frac{\partial X}{\partial\theta} \\
&- \quad 2X^2Z^2\frac{\partial\gamma}{\partial\theta}\frac{\partial Y}{\partial\theta} - 2X^2YZ\frac{\partial\gamma}{\partial\theta}\frac{\partial Z}{\partial\theta} + 4XY^2Z^2\frac{\partial^2\gamma}{\partial r^2} - 2Y^2Z^2\frac{\partial\gamma}{\partial r}\frac{\partial X}{\partial r} \\
&+ \quad 2XYZ^2\frac{\partial\gamma}{\partial r}\frac{\partial Y}{\partial r} + 2XY^2Z\frac{\partial\gamma}{\partial r}\frac{\partial Z}{\partial r} - 2XYZ^2\frac{\partial^2 X}{\partial\theta^2} + YZ^2\left[\frac{\partial X}{\partial\theta}\right]^2 \\
&+ \quad XZ^2\frac{\partial X}{\partial\theta}\frac{\partial Y}{\partial\theta} + YZ^2\frac{\partial X}{\partial r}\frac{\partial Y}{\partial r} + X^2Z\frac{\partial Y}{\partial\theta}\frac{\partial Z}{\partial\theta} - 2XYZ^2\frac{\partial^2 Y}{\partial r^2} \\
&+ \quad XZ^2\left[\frac{\partial Y}{\partial r}\right]^2 - XYZ\frac{\partial Y}{\partial r}\frac{\partial Z}{\partial r} - 2X^2YZ\frac{\partial^2 Z}{\partial\theta^2} + X^2Y\left[\frac{\partial Z}{\partial\theta}\right]^2\Bigg) \\
R_{23} &= 0 \\
R_{33} &= \frac{e^{2\gamma}}{4X^2Y^2Z}\Bigg(-4X^2YZ^2\frac{\partial^2\psi}{\partial\theta^2} - 2XYZ^2\frac{\partial\psi}{\partial\theta}\frac{\partial X}{\partial\theta} + 2X^2Z^2\frac{\partial\psi}{\partial\theta}\frac{\partial Y}{\partial\theta} \\
&- \quad 2X^2YZ\frac{\partial\psi}{\partial\theta}\frac{\partial Z}{\partial\theta} - 4XY^2Z^2\frac{\partial^2\psi}{\partial r^2} + 2Y^2Z^2\frac{\partial\psi}{\partial r}\frac{\partial X}{\partial r} - 2XYZ^2\frac{\partial\psi}{\partial r}\frac{\partial Y}{\partial r} \\
&- \quad 2XY^2Z\frac{\partial\psi}{\partial r}\frac{\partial Z}{\partial r} - XYZ\frac{\partial X}{\partial\theta}\frac{\partial Z}{\partial\theta} + Y^2Z\frac{\partial X}{\partial r}\frac{\partial Z}{\partial r} + X^2Z\frac{\partial Y}{\partial\theta}\frac{\partial Z}{\partial\theta} \\
&- \quad XYZ\frac{\partial Y}{\partial r}\frac{\partial Z}{\partial r} - 2X^2YZ\frac{\partial^2 Z}{\partial\theta^2} + X^2Y\left[\frac{\partial Z}{\partial\theta}\right]^2 - 2XY^2Z\frac{\partial^2 Z}{\partial r^2} \\
&+ \quad XY^2\left[\frac{\partial Z}{\partial r}\right]^2\Bigg).
\end{aligned}
$$