Image based vision correction: a study of total variation methods

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January 17, 2013
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Create an algorithm that corrects an image based on a diopter value predetermined by the user. So that the corrected version of the image has better optical properties than the original image.
Blurring Model

Denote by $d(x, y)$ the original data of the image, $k(x, x', y, x')$ the blurring kernel of the eye and $b(x, y)$ the resulting blurred image then

$$b(x, y) = \int_{\Omega} k(x, x', y, y')d(x', y')dx'dy'$$

where $\Omega$ is the domain of image. The kernel $k$ is also called the point spread function (PSF).
Spatial Invariance

Under spatial invariance of the PSF the model become the convolution operator,

\[ b(x, y) = \int_{\Omega} k(x - x', y - y')d(x', y')dx'dy' = k \ast d(x, y). \]
Since convolution becomes multiplication in the Fourier domain we have by the inversion formula of Fourier transform

\[ d(x, y) = \frac{1}{2\pi} \int e^{iz \cdot \xi} \hat{d}(\xi) \hat{k}(\xi) d\xi, \]

where \( \xi \) is the dual variable of \( z = (x, y) \) and \( \hat{d}(\xi), \hat{k}(\xi) \) denotes the Fourier transform of \( d, k \) respectively.

\(^\dagger\) Under some condition of the image.
Eigenvalues and eigenvectors of the blurring kernel:
Ill-Conditioned Problem

- **Forward problem.**
  1. Losses higher frequencies (weights decay).
  2. Stable (small perturbation generates small change in the outcome).

- **Inverse Problem**
  1. Very unstable
  2. Need for regularization.
Correction Model

Denote by $d(x, y)$ the original data of the image, $u(x, y)$ the corrected image and by $k(x - x', y - y')$ the blurring kernel of the eye, then the model becomes

$$d(x, y) = \int_{\Omega} k(x - x', y - y')u(x', y')dx' dy' = k * u(x, y).$$

$$b(x, y) = \int_{\Omega} k(x - x', y - y')d(x', y')dx' dy' = k * d(x, y).$$

Remark: Change of perspective for same equation.
Same Problem?

• Positive answer:
  1. Shares the operators properties (ill-conditioned).
  2. Regularization could improve results.

• Negative answer:
  1. Requires “non trivial” constraints.
  2. No unique solution.
Constraint Conditions

Figure 1: Completely changes the geometry of the problem
To get better stability for the problem we use regularization to impose additional characteristics to the required solution

\[ d^* = \min_d \left\{ \frac{1}{2} \int_{\Omega} |k \ast d - b|^2 dz + \alpha \int_{\Omega} |\nabla d|^2 dz \right\}. \]

In a discrete setting

\[ d^* = \min_d \left\{ \frac{1}{2} \|Kd - b\|_2^2 + \alpha \|d\|_2^2 \right\}. \]
Total Variation Regularization

In case of “block type” images a very successful method first implemented by Rudin, Osher and Fatemi is the total variation method.

\[ d^* = \min_u \{ \frac{1}{2} \int_\Omega (k * d - b)dz + \alpha \int_\Omega |\nabla d|dz \}. \]

In discrete setting

\[ d^* = \min_d \{ \frac{1}{2} ||Kd - b||_2^2 + \alpha ||\nabla d||_1^2 \}. \]

Why \( l^1 \) norm is better than \( l^2 \) norm?
Figure 2: Original and blurred images.
Comparison of Tikhonov and TV

Reconstruction on the left were obtained using identity penalty matrix. Those on the right were obtained using negative Laplacian. From top to bottom: $\alpha = 10^{-2}, \alpha = 10^{-4}, \alpha = 10^{-6}$
Figure 3: Total variation reconstruction
Lagged Diffusivity Method

Euler-Lagrange equations of $d^* = \frac{1}{2} \| Kd - b \|_2^2 + \| \nabla d \|_1^2$

$$K^*(Kd - b) - \alpha \nabla \cdot \left( \frac{\nabla d}{|\nabla d|} \right) = 0.$$  

There is a problem in the regions that we are trying to approximate ($|\nabla d| = 0$).
We use the approximation

$$\psi(t) = \sqrt{t^2 + \beta} \sim |t|$$

when $\beta$ is small.
Approximation of Total Variation

Introduce $\psi(t) = \sqrt{t^2 + \beta}$,

$$d^* = \min_u \left\{ \frac{1}{2} \int_\Omega (k \ast d - b) dz + \alpha \int_\Omega \sqrt{|\nabla d|^2 + \beta^2} dz \right\}.$$  

In discrete setting

$$d^* = \min_d \left\{ \frac{1}{2} \| Kd - b \|_2^2 + \alpha \| \sqrt{|\nabla d|^2 + \beta^2} \|_1 \right\} .$$
With this modification the Gateux derivative becomes,
\[ K^* (Kd - b) - \alpha \nabla \cdot \left( \frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta^2}} \right) = 0. \]

Using fixed point iteration by Vogel and Oman
\[ K^* (Kd^{k+1} - b) - \alpha \nabla \cdot \left( \frac{\nabla d^{k+1}}{\sqrt{|\nabla d^k|^2 + \beta^2}} \right) = 0. \]
Lagged Diffusivity Method

De-blurring Image: \( b(x,y)=\int_{\Omega} k(x-x',y-y')d(x',y')dx'dy' \).

\[
d^* = \min_d \left\{ \frac{1}{2} \int_{\Omega} (k*d - b)dz + \alpha \int_{\Omega} \sqrt{|\nabla d|^2 + \beta^2}dz \right\}.
\]

\[
K^*(Kd - b) - \alpha \nabla \cdot \left( \frac{\nabla d}{\sqrt{|\nabla d|^2 + \beta^2}} \right) = 0.
\]

Correction Image: \( d(x,y)=\int_{\Omega} k(x-x',y-y')u(x',y')dx'dy' \).

\[
u^* = \min_u \left\{ \frac{1}{2} \int_{\Omega} (k*u - d)dz + \alpha \int_{\Omega} \sqrt{|\nabla u|^2 + \beta^2}dz \right\}.
\]

\[
K^*(Ku - d) - \alpha \nabla \cdot \left( \frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta^2}} \right) = 0.
\]
De-blurring Image: \( b(x,y) = \int_{\Omega} k(x-x',y-y') d(x',y') dx' dy' \).

\[
d^* = \min_d \{ \frac{1}{2} \int_{\Omega} (k \ast d - b) dz + \alpha \int_{\Omega} \sqrt{\left| \nabla d \right|^2 + \beta^2} dz \}.
\]

\[
K^* (Kd - b) - \alpha \nabla \cdot \left( \frac{\nabla d}{\sqrt{\left| \nabla d \right|^2 + \beta^2}} \right) = 0.
\]

Correction Image: \( d(x,y) = \int_{\Omega} k(x-x',y-y') u(x',y') dx' dy' \).

\[
u^* = \min_u \{ \frac{1}{2} \int_{\Omega} (k \ast u - d) dz + \alpha \int_{\Omega} \sqrt{\left| \nabla k \ast u \right|^2 + \beta^2} dz \}.
\]

\[
K^* (Ku - d) - \alpha K^* \nabla \cdot \left( \frac{\nabla Ku}{\sqrt{\left| \nabla Ku \right|^2 + \beta^2}} \right) = 0.
\]
Using fixed point iteration by Vogel and Oman

\[ K^* (Ku^{k+1} - b) - \alpha K^* \nabla \cdot \left( \frac{\nabla u^{k+1}}{\sqrt{|\nabla K u^k|^2 + \beta^2}} \right) = 0. \]

Remarks: The main advantage of this method is that it has global convergence, but it has linear ratio of convergence.
Future Work

- Prime-Dual Method.
- Constraint problem.
- Contrast.
- Local behavior.